

# Modeling Dissipation Equation in Supersonic Turbulent Mixing Layers with High-Density Gradients

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The ability of a  $k$ - $\varepsilon$  model to compute mixing layers with significant density gradients is discussed. It is shown that, as in boundary layers, deficiencies appear for compressible flows, even if the  $k$ - $\varepsilon$  model is improved by compressibility corrections. A modification of the  $\sigma_\varepsilon$  coefficient is proposed to compute efficiently compressible mixing layers with high-density variations. This new formulation leads to a separate modeling of the subsonic and supersonic effects. The results are presented on mixing layer computations with density ratios between  $\frac{1}{7}$  (velocity/density cogradients) and 7 (velocity/density cross gradients), and the convective Mach number evolves from 0.3 to 1. They show a clear improvement of the capability of the  $k$ - $\varepsilon$  model to compute accurately high-speed mixing layers.

## Nomenclature

$a$	= local speed of sound
$C_p$	= specific heat at constant pressure
$C_{\varepsilon_1}$	= $\varepsilon$ -equation coefficient, 1.44
$C_{\varepsilon_2}$	= $\varepsilon$ -equation coefficient, 1.92
$E$	= mean total energy
$k$	= kinetic energy of turbulence
$M$	= Mach number
$M_c$	= convective Mach number, $(u_1 - u_2)/(a_1 + a_2)$
$M_t$	= turbulent Mach number, $\sqrt{(2k)/a}$
$\mathcal{P}$	= turbulence production term
$Pr$	= laminar Prandtl number, 0.72
$Pr_t$	= turbulent Prandtl number, 0.7
$p$	= mean static pressure (Reynolds average)
$q$	= coefficient of proportionality (between $\mathcal{P}$ and $\varepsilon_t$ )
$r$	= velocity ratio, $u_2/u_1$
$S_{ij}$	= mean rate of strain tensor
$s$	= density ratio, $\rho_2/\rho_1$
$T$	= mean static temperature (Reynolds average)
$U_c$	= convection velocity of large structures
$u_i$	= Favre-averaged velocity components ( $u, v$ )
$x$	= horizontal coordinate
$y$	= vertical coordinate
$\Delta U$	= velocity difference between the external streams
$\delta$	= mixing layer thickness
$\delta'$	= compressible spreading rate
$\delta'_0$	= incompressible spreading rate
$\varepsilon$	= solenoidal dissipation rate
$\varepsilon_c$	= compressible dissipation rate
$\varepsilon_t$	= total dissipation rate, $\varepsilon + \varepsilon_c$
$\lambda$	= molecular conductivity coefficient

$\lambda_t$	= turbulent conductivity coefficient
$\mu$	= dynamic viscosity
$\mu_t$	= eddy viscosity
$\nu$	= kinematic viscosity, $\mu/\rho$
$\rho$	= mean density (Reynolds average)
$\Phi$	= dimensionless rate of spread

## Subscripts

1, 2	= faster and slower flows
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## I. Introduction

IT is well known that the standard  $k$ - $\varepsilon$  model fails to predict highly compressible flows, such as supersonic mixing layers. In such high-speed flows, from a modeling point of view, two aspects have to be considered: a density variation and a Mach number effect.

In the early 1990s some models, called compressibility corrections,<sup>1-3</sup> were developed to cure the deficiencies due to Mach number effects, appearing mainly in the poor prediction of the decrease of the spreading rate of compressible mixing layers with standard models. Compressibility can act in several ways in turbulence modeling: It can change the dissipation rate and the pressure dilatation term in the  $k$ - $\varepsilon$  equation. It may alter the anisotropy of the Reynolds stress and the diffusion rate of kinetic energy. First-generation models have underlined the possible importance of the first two effects. Recent developments suggest that other mechanisms can be involved. For instance, the compressibility correction models do not take into account the density gradient influence and usually have been calibrated on uniform density test cases. The present paper is focused on density gradient effects combined with the influence of the Mach number.

Works on density variation effects in supersonic flows are still recent and have been essentially focused on boundary layers.<sup>4,5</sup> A major contribution of Ref. 4 was to show that the  $k$ - $\varepsilon$  model fails to reproduce the logarithmic law because of density variations. The analysis of Ref. 4 concluded that the  $\varepsilon$  equation is responsible for this problem.

The cure proposed in Ref. 4 is to account for density gradients through coefficients  $\sigma_\varepsilon$  or  $C_{\varepsilon_1}$ . Following the same idea, Aupoix and Viala<sup>5</sup> suggested modifying the diffusion term of the  $\varepsilon$  equation, so that density gradients are counterbalanced.

The present paper examines to what extent such ideas can be applied to free shear flows such as mixing layers. The initial choice was to use a  $k$ - $\varepsilon$  type model. Rather than presenting a complete,

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ready to use model of turbulence, the objective of this work is to recall, in a simple way, that standard models have difficulties in correctly predicting mixing layers with strong density gradients. This drawback, although minor in low-speed flows is of importance at high speeds when compressibility models are used. Considering the properties of turbulence near the center of a self-similar mixing layer, it will be shown that the relation between  $\sigma_\varepsilon$  and the other coefficients of the model is not the same as in the equilibrium zone of a boundary layer. A relation independent of the models will be proposed, in which the relative importance of the phenomena involving Mach number and density gradients appear clearly. To illustrate the importance of this question, and as the models are rather sensitive to the values of the constants, simple requirements from empirical origin will be proposed to find necessary conditions for turbulent closure to give a correct prediction of simple flows, such as supersonic free shear layers.

## II. Governing Equations and the $k$ - $\varepsilon$ Model

For turbulent flows, governing equations are obtained from the averaged Navier-Stokes equations. The modeling of compressible flow equations is often made with two kinds of averages: velocity components  $u_i$ , temperature  $T$ , and total energy  $E$  are Favre mass-weighted averaged whereas a Reynolds average is used for density  $\rho$  and pressure  $p$ . To express the unknown correlations, a turbulence model is required. The closure for Reynolds stresses is given by Boussinesq's assumption:  $-\rho u_i'' u_j'' = \mu_t S_{ij} - 2/3 \rho k \delta_{ij}$  with  $\mu_t = C_\mu (\rho k^2 / \varepsilon)$  the eddy viscosity,  $k = \frac{1}{2} \overline{u_i'' u_i''}$  the turbulent kinetic energy, and  $S_{ij} = \partial u_i / \partial x_j + \partial u_j / \partial x_i - \frac{2}{3} (\partial u_k / \partial x_k) \delta_{ij}$  the rate of strain. The use of the  $k$ - $\varepsilon$  model<sup>6</sup> results in the following set of equations:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} &= 0 \\ \frac{\partial \rho u_j}{\partial t} + \frac{\partial (\rho u_i u_j + p \delta_{ij})}{\partial x_i} &= \frac{\partial}{\partial x_i} \left[ (\mu + \mu_t) S_{ij} - \frac{2}{3} \rho k \delta_{ij} \right] \\ \frac{\partial E}{\partial t} + \frac{\partial u_i (E + p)}{\partial x_i} &= \frac{\partial}{\partial x_i} \left[ (\mu + \mu_t) S_{ij} u_j + (\lambda + \lambda_t) \frac{\partial T}{\partial x_i} + \frac{\mu_t}{\sigma_k} \frac{\partial k}{\partial x_i} - \frac{2}{3} \rho k u_i \right] \\ \frac{\partial \rho k}{\partial t} + \frac{\partial \rho u_i k}{\partial x_i} &= \frac{\partial}{\partial x_i} \left[ \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_i} \right] + \mathcal{P} - \rho \varepsilon \\ \frac{\partial \rho \varepsilon}{\partial t} + \frac{\partial \rho u_i \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i} \left[ \left( \mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_i} \right] + C_{\varepsilon 1} \frac{\varepsilon}{k} \mathcal{P} - C_{\varepsilon 2} \rho \frac{\varepsilon^2}{k} \quad (1) \end{aligned}$$

with  $\mathcal{P} = -\rho \overline{u_i'' u_j''} (\partial u_i / \partial x_j)$ , and the production term of  $k$  and the total energy  $E$  defined as  $E = \rho (C_v T + 1/2 u_i u_i + k)$ . The model is quite standard in compressible turbulence modeling. For details and explanations concerning this model, the reader can refer to the survey article of Gatski.<sup>7</sup>

## III. Density Gradients Modeling

### A. $\varepsilon$ Equation

According to Huang et al.,<sup>4</sup> the current modeling of the  $\varepsilon$  equation presents an incompatibility with the existence of high-density gradients. The reason is that the standard  $k$ - $\varepsilon$  model uses a set of modeling constants that remains unchanged whatever the flow. The usual values of these constants, as they are employed in the present work, are as follows:

$$\begin{aligned} C_{\varepsilon 1} &= 1.44, & C_{\varepsilon 2} &= 1.92, & \sigma_\varepsilon &= 1.3 \\ \sigma_k &= 1.0, & C_\mu &= 0.09 \end{aligned}$$

For constant density flows, the local equilibrium condition in the wall region of a boundary layer leads to the following relation:

$$[\sigma_\varepsilon (C_{\varepsilon 2} - c_{\varepsilon 1}) / \kappa^2] \sqrt{C_\mu} = 1 \quad (2)$$

For variable density boundary layers, relation (2) is not verified, and density gradients have to be taken into account. If gradients of

$\sigma_\varepsilon$  and if the diffusion of  $k$  are negligible, the alternate following relation can be derived:

$$\frac{\sigma_\varepsilon (C_{\varepsilon 2} - C_{\varepsilon 1})}{\kappa^2} \sqrt{C_\mu} = 1 + F \left( \frac{y}{\rho} \frac{\partial \rho}{\partial y}, \frac{y^2}{\rho} \frac{\partial^2 \rho}{\partial y^2} \right)$$

showing that density gradient effects enter into the compatibility relationship between modeling constants. According to Huang et al.,<sup>4</sup> this explains the difficulties of the  $k$ - $\varepsilon$  model to reproduce the logarithmic law of a compressible boundary layer. In such flows, the term  $F(y/\rho \partial \rho / \partial y, y^2/\rho \partial^2 \rho / \partial y^2)$  cannot be neglected. The same idea is used in the present work to study mixing layers with large density variations. However, self-similar mixing layers do not obey equilibrium conditions, diffusion cannot be neglected, and  $\varepsilon$  varies along the flow. Therefore, it may be suspected that the analysis of wall flows is not valid for free shear flows. This is examined in the next section.

### B. Description of the Density Gradient and Compressible Turbulence Corrections

Our aim is to propose a relation between the  $\varepsilon$ -equation coefficients to describe properly incompressible mixing layers with density gradients and then to determine if such new values are compatible with the existing compressibility models. We chose to keep constant  $C_\mu$ ,  $C_{\varepsilon 1}$ ,  $C_{\varepsilon 2}$ , and  $\sigma_k$  and to express  $\sigma_\varepsilon$  in terms of other coefficients to obtain a relation that takes into account subsonic effects (density variations) and supersonic effects (compressibility) altogether. We consider the following  $k$  and  $\varepsilon$  equations:

$$\begin{aligned} \rho \frac{Dk}{Dt} &= \frac{\partial}{\partial y} \left( \frac{\mu_t}{\sigma_k} \frac{\partial k}{\partial y} \right) + \mathcal{P} - \rho \varepsilon_t + \overline{p'd'} \\ \rho \frac{D\varepsilon}{Dt} &= \frac{\partial}{\partial y} \left( \frac{\mu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial y} \right) + C_{\varepsilon 1} \frac{\varepsilon}{k} \mathcal{P} - C_{\varepsilon 2} \frac{\varepsilon}{k} \rho \varepsilon \quad (3) \end{aligned}$$

where  $\overline{p'd'}$  is the pressure-dilatation term and  $\varepsilon_t$  is the total dissipation.

Early compressible models<sup>1-3</sup> include explicit terms of compressibility such as the pressure dilatation correlation and the dilatation dissipation term  $[\varepsilon_c = 4/3 \mu (\nabla \cdot u'')^2]$ , which is part of the total dissipation rate. Recent developments in compressible models<sup>8,9</sup> suggest that the dilatation dissipation varies like  $M_t^4$ , and, therefore, in many supersonic turbulent flows where  $M_t \ll 1$ , this contribution to dissipation may be neglected. For weak compressibility, the pressure dilatation term varies like  $M_t^2$ , but is negligible when production balances dissipation. A consequence is that in many situations, this term may be weak too. The main effect of compressibility, as suggested by the numerical simulations of Sarkar<sup>10</sup> and of Lele et al.,<sup>11</sup> therefore, can be a decrease of the Reynolds stress tensor isotropy. Note that the experimental attempts to check this hypothesis are technically difficult; there is no clear indication of changes in the anisotropy in Refs. 12 and 13, whereas the measurements in Refs. 14 and 15 suggest that such an effect is important. Models representing this effect are not yet available. The physical guesses used by the existing first-generation models are certainly questionable. However, because they were tuned to compute supersonic mixing layers with constant density, they have probably some ability to represent globally the effect of compressibility on turbulence in mixing layers. More precisely, it may be shown in a simple case how an effect attributed to dissipation may represent an alteration of production. For example, consider the equilibrium zone of a compressible boundary layer in which

$$\mathcal{P} = \varepsilon_t \quad \text{or} \quad -\overline{u'' v''} \frac{\partial U}{\partial y} = \varepsilon + \varepsilon_c$$

If we express the dilatation dissipation in the form  $\varepsilon_c = \varepsilon f(M_t)$ , the preceding equality can be rewritten as

$$\sqrt{C_\mu} k \frac{\partial U}{\partial y} \left[ 1 - f(M_t) \frac{\varepsilon}{\mathcal{P}} \right] = \varepsilon$$

If  $\varepsilon/\mathcal{P}$  is not far from 1, the final relation shows that  $f(M_t)$ , which was interpreted as a compressible dissipation correction, can be understood as a correction to  $C_\mu$ . In mixing layers, the equilibrium condition is not fulfilled, even at low speed. However, it is found from Wygnanski and Fiedler's work<sup>16</sup> that  $\varepsilon = 0.7$  or  $0.8\mathcal{P}$ . Assuming that the proportionality constant is not dramatically altered by compressibility, the same analysis can be used, at least qualitatively. In the examples presented here, existing models although questionable in their interpretation will be used to explore their consequences in the calculations. They involve the pressure-dilatation term  $\overline{p'd'}$  and the total dissipation  $\varepsilon_t$ . According to Sarkar et al.<sup>1</sup> or Zeman,<sup>2</sup> this term can be decomposed into a solenoidal part (notated  $\varepsilon_s$ , which verifies the earlier  $\varepsilon$  equation) and a compressible part (which needs to be modeled). Many models (for example, those of Refs. 1–3) have the form  $\varepsilon_t = F(M_t) \times \varepsilon$ , with  $F(M_t) = 1 + \text{function}(M_t)$ . Moreover, the definition of the eddy viscosity has to be reconsidered. The usual low-speed form is  $C_\mu \rho k^2 / \varepsilon$ . The value of  $C_\mu$ , identified as  $(-u''v''/k)^2$ , derives from equilibrium considerations. The equilibrium can now be expressed as  $\mathcal{P} = \varepsilon_t$ , in which  $\varepsilon_t = \varepsilon + \varepsilon_c$ . This modification leads to the relation

$$v_t = C_\mu \rho k^2 / (\varepsilon + \varepsilon_c)$$

This shows that the total dissipation should be used in this relation if the relation  $C_\mu = (-u''v''/k)^2$  is still used. If dilatation-dissipation is small, this refinement makes no large difference; however, with the existing models,  $\varepsilon_c$  can have significant values, and some caution should be taken.

To check the generality of relationships between the constants of the models, self-similar mixing layers in the vicinity of the maximum of turbulent kinetic energy are considered. The main idea is to find conditions to ensure that a correct balance in the equation for  $\varepsilon$  in the zone where the terms are significant, namely, in the center of the layer. It is recalled that, in such flows, self-similarity implies that the peak value of turbulent kinetic energy is constant along the flow and that the locus of the maximum of  $k$  is a streamline. Moreover, streamlines are straight lines. Choosing this particular streamline as longitudinal axis of coordinate implies

$$\frac{Dk}{Dt} = 0, \quad \frac{\partial k}{\partial y} = 0 \quad \text{for} \quad y = 0$$

In general, it is found in experiments (for example, see Ref. 16 for a subsonic reference) that, within measurement accuracy,  $v_t$  is approximately constant at the vicinity of the maximum of  $k$ . This approximation will be used in the present derivation. It implies that the maximum of friction is located at the inflection point of the mean velocity profile.

As suggested in Ref. 15, and although dissipation measurements are difficult, it is assumed that in the center of the mixing layer production and dissipation are proportional. With such an assumption, considering the definition of  $v_t = C_\mu k^2 / \varepsilon$  and  $\varepsilon = q\mathcal{P}$ , it is straightforward to show that  $(\tau/k)^2 = C_\mu / q$  where  $\tau = -u''v''$  is the turbulent friction.

Therefore, proportionality of production and dissipation implies that the maximum of friction  $\partial\tau/\partial y$  is located on the axis where  $\partial k/\partial y = 0$ . The hypothesis of a weak gradient of  $v_t$  implies that the maximum of  $\varepsilon$  occurs at the same point, which is also the inflection point of the mean velocity profile.

Injecting these properties into Eq. (3) yields

$$\begin{aligned} \frac{\mu_t}{\sigma_k} \frac{\partial^2 k}{\partial y^2} + \mathcal{P} - \rho \varepsilon_t + \overline{p'd'} &= 0 \\ \rho \frac{D\varepsilon}{Dt} &= \frac{\mu_t}{\sigma_\varepsilon} \frac{\partial^2 \varepsilon}{\partial y^2} + C_{\varepsilon_1} \frac{\varepsilon}{k} \mathcal{P} - C_{\varepsilon_2} \frac{\varepsilon}{k} \rho \varepsilon \end{aligned} \quad (4)$$

If  $\mathcal{P}$  and  $\varepsilon$  are proportional when self-similarity is achieved, it is clear that  $\varepsilon$  should decrease along  $x$ . The derivative  $D\varepsilon/Dt$  is approximated by

$$u \frac{\partial \varepsilon}{\partial x} \sim U_c \frac{\partial}{\partial x} q \mathcal{P}$$

For further simplification, it was convenient to use the convection velocity  $U_c$  instead of the local mean velocity  $u$ . As a matter of fact, in many cases of supersonic mixing layers,  $U_c$  remains close to the local velocity in the middle of the layer. From the definition of the vorticity thickness  $\delta$ , here  $\partial U/\partial y = \Delta U/\delta$  along the streamline considered, and using the relation proposed by Papamoschou and Roshko<sup>17</sup> for the spreading rate  $d\delta/dx$

$$\frac{d\delta}{dx} = \frac{1}{2} \delta_{\text{ref}}' \frac{\Delta U}{U_c} \Phi(M_c)$$

where  $\delta_{\text{ref}}'$  is the spreading rate of the subsonic half-jet with constant density, it is easy to show that

$$U \frac{\partial \varepsilon}{\partial x} \sim -\frac{1}{2} \varepsilon \frac{\partial U}{\partial y} \delta_{\text{ref}}' \Phi(M_c)$$

By introducing these approximations into Eq. (4) and eliminating the diffusion of turbulent energy between the two equations, the following relation is obtained:

$$\sigma_\varepsilon = \frac{\sigma_k(q-1) - \frac{\sigma_k}{\tau/k} \frac{\overline{p'd'}}{\rho k} \left(\frac{\partial U}{\partial y}\right)^{-1} + v_t \frac{\partial^3 U}{\partial y^3} \left(\frac{\partial U}{\partial y}\right)^{-2} \frac{k}{\tau}}{\frac{q}{F(M_t)} C_{\varepsilon_2} - C_{\varepsilon_1} - \frac{1}{2} \delta_{\text{ref}}' \frac{\Phi(M_c)}{\tau/k}}$$

A first important difference with the case of boundary layers can be seen: The density gradient does not appear explicitly. The reason is that the term  $\partial\rho/\partial y$  is associated with  $\partial k/\partial y$  or  $\partial\varepsilon/\partial y$ , which are zero along the considered streamline. Two other elements are present. First, the term of the numerator involving  $\sigma_k$  results from the diffusion term of the  $k$  equation. It vanishes when  $q = 1$ , that is,  $\mathcal{P} = \varepsilon$ . Second, the last term of the denominator originates from the longitudinal variations of dissipation. The consequence is that the relationship valid in the log zone of a boundary layer cannot be suitable for the center of a mixing layer, where the diffusion term is maximum. It can be seen that compressibility can act explicitly through pressure-dilatation,  $F$  and  $\Phi$ , and implicitly through the influence of Mach number on  $v_t$ . It is clear that many parameters have to be taken into account. This expression is strictly valid at the vicinity of the maximum of  $k$ . However, we wanted to ensure that a correct balance in the equation for  $\varepsilon$  is obtained in the part of the flow where the different source terms have significant values. For this reason, conditions in the center of the layer were defined. It is expected that the resulting formula represents an acceptable assumption over the central third or half of the layer. It is certainly not valid near the edge, where gradients of density, of  $k$  and of  $\varepsilon$ , are present. However, as the overall level of the source terms is weaker in this zone, the consequence of using a relation with constant  $\sigma_\varepsilon$  is probably of minor importance. A constraint forcing the model to reproduce the correct level of dissipation and kinetic energy was found by imposing an empirical input to specify  $v_t$ . Following Ref. 18, the eddy viscosity can be expressed as a function of the mixing layer parameters:

$$v_t = \frac{1}{2} K(s, r, M_c, M_g) \delta_{\text{ref}}' \Delta u \delta \Phi(M_c)$$

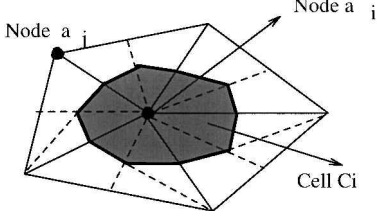
Replacing the velocity gradient by  $\Delta U/\delta$  and expressing the third derivative as  $\partial^3 u/\partial y^3 = f(s, r, M_c, M_g)(\Delta u/\delta^3)$ , the final expression, which will be used in the examples presented in a next section, is obtained:

$$\sigma_\varepsilon = 2F \frac{\sigma_k(q-1) \frac{\tau}{k} - \sigma_k \frac{\overline{p'd'}}{\rho k} \frac{\delta}{\Delta U} + \frac{\delta_{\text{ref}}' \Phi K f}{2}}{2q \frac{\tau}{k} C_{\varepsilon_2} - F \left( 2 \frac{\tau}{k} C_{\varepsilon_1} + \delta_{\text{ref}}' \Phi \right)} \quad (5)$$

In expression (5),  $F(M_t)$  and  $\overline{p'd'}$  depend on the choice of the compressibility model, so that there are only two free parameters left:  $q$  and the product  $Kf$ . For subsonic flows, the experiments of Wygnanski and Fiedler<sup>16</sup> suggest for  $q$  a value close to 0.7. In the present work, it was found for incompressible mixing layers

**Table 1**  $Kf$  and corresponding  $\sigma_\varepsilon$  values vs  $s$ 

$s$	$\frac{1}{7}$	1	7
$Kf$	-2.5	-0.225	0.35
$\sigma_\varepsilon$	4.33	1.3	0.53

**Fig. 1** Control cell  $C_i$ .

that a value of 0.75 is well fitted to  $k$ - $\varepsilon$  calculations. This value was kept for compressible calculations.  $Kf = K(s, r, M_c, M_g) \times f(s, r, M_c, M_g)$  contains the influence of the density on  $\sigma_\varepsilon$ , through the parameter  $s$ . From the analysis of self-similar mixing layers and from experiments,<sup>19,20</sup> it is reasonable to assume for  $M_c \leq 1$  that the product  $Kf$  is almost constant with the convective Mach number. As a consequence, the following procedure has been retained. First, incompressible case computations (Sec. V.B) are used to determine acceptable values of  $\sigma_\varepsilon$  to predict the correct spreading rate. The values of  $Kf$  are deduced (see Table 1). Second, compressible case computations (Sec. V.C) are performed. The compressibility correction proposed in Ref. 21 is considered:  $\varepsilon_c = \alpha M_i^2$ , where  $\alpha = 0.5$ , with  $F(M_i) = 1 + \alpha M_i^2$  and  $p'd' = 0$ . The value of  $\alpha$  has been calibrated on a mixing layer with uniform density ( $s = 1$ ). It will be shown that this value has to be reconsidered for flows with variable density ( $s \gg 1$  or  $s \ll 1$ ).

#### IV. Numerical Method

In this study, the numerical method is based on a mixed finite element/finite volume spatial approximation, and the MUSCL technique is used to obtain a second-order spatial accuracy. Equations (1) can be rewritten in divergence form as

$$\frac{\partial \mathbf{W}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{W}) = \nabla \cdot \mathbf{R}(\mathbf{W}) + \nabla \cdot \mathbf{S}(\mathbf{W}) + \Omega(\mathbf{W})$$

where  $\mathbf{W} = (\rho, \rho u, \rho v, E', \rho k, \rho \varepsilon)'$  the vector of the conservative variables and  $E'$  denotes the modified total energy:

$$E' = p/(\gamma - 1) + \frac{1}{2}\rho(u^2 + v^2) \quad \text{with} \quad p' = p + \frac{2}{3}\rho k$$

$$E = p/(\gamma - 1) + \frac{1}{2}\rho(u^2 + v^2) + \rho k$$

$$\Rightarrow E' = E + \beta \rho k \quad \text{with} \quad \beta = -1 + [2/3(\gamma - 1)]$$

and where  $\mathbf{F}$  are the hyperbolic flux functions,  $\mathbf{R}$  and  $\mathbf{S}$  are the laminar and turbulent diffusive flux functions, and  $\Omega$  is the source term due to the  $k$ - $\varepsilon$  model.

The discrete equations are obtained from a finite volume scheme applied to hyperbolic fluxes and a finite element method applied to the diffusive and source terms. For the finite element method, the terms are evaluated on each triangle with a classical P1-Galerkin formulation. For the finite volume method, the fluxes are calculated at the interface of two cells (Fig. 1). A Roe's approximate Riemann solver is used to integrate the mean flow variables ( $\rho$ ,  $\rho u$ ,  $\rho v$ , and  $E'$ ), whereas a multicomponents flux as proposed by Larrourou<sup>22</sup> is used for the turbulent ones ( $\rho k$  and  $\rho \varepsilon$ ), to preserve positivity.

Such fluxes have a first-order spatial accuracy, and a second-order scheme is obtained with the MUSCL approach. The results were obtained on a 4840 point rectangular mesh. Open boundary conditions are approximated by the Steger-Warming flux<sup>23</sup> in the inlet and outlet sections. A slip condition ( $\mathbf{V} \cdot \mathbf{n} = 0$ ) is applied to the superior and inferior boundaries of the computational domain. An implicit first-order, preconditioning, time-advancing method is employed, and the steady state is considered as being reached when the residual on total energy normalized by the first-step value is  $10^{-6}$ . Technical details on the numerical method employed in this

study may be found in Ref. 24 or 25. For each test case, it has been verified that the computational domain has a mesh refined enough to have a solution independent of the discretization step.

## V. Results and Discussion

### A. Validation Test Cases

Essentially two kinds of models are examined: the standard  $k$ - $\varepsilon$  model (with  $\sigma_\varepsilon = 1.3$ ) and its modified version with variable  $\sigma_\varepsilon$  [relation (5)]. They are applied to mixing layers with large density gradients, in subsonic and supersonic conditions. The investigated convective Mach numbers range from values close to 0 to near 1. The validation test cases include velocity/density cogradients and cross gradients ( $s = \frac{1}{7}$ , 1, and 7). Furthermore, the same velocity ratio (a moderate value of  $r = 0.65$ ) was chosen for all of the studied flows. Characteristics of the flows are given in Table 2.

### B. Incompressible Mixing Layer Results

The aim of this section is to obtain an adaptation of the  $k$ - $\varepsilon$  model to low-speed mixing layers with variable density. Relation (5), in which  $Kf$  is an unknown will be examined. If we consider the peculiarities of the present test cases, expression (5) can be simplified by noting that, in this case,  $p'd' = 0$ ,  $F(M_i) = 1$ , and  $\Phi(M_c) = 1$ . Under these conditions, relation (5) has been estimated by

$$\sigma_\varepsilon = 1 - \frac{4}{3}Kf \quad (6)$$

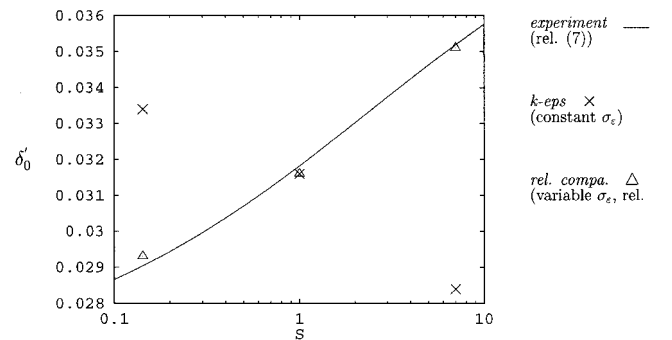
The spreading rate  $\delta_{0\text{exp}}$  is given by the following semiempirical relation<sup>17</sup>:

$$\delta_{0\text{exp}} = \frac{0.15}{2} \times \frac{(1-r)(1+\sqrt{s})}{(1+r \times \sqrt{s})} \quad (7)$$

In Fig. 2, the corresponding curve is referenced by experiment; the results of the standard  $k$ - $\varepsilon$  model with constant  $\sigma_\varepsilon$  ( $\sigma_\varepsilon = 1.3$ ) are denoted by  $k$ -eps, and rel. compa. refers to computations with variable  $\sigma_\varepsilon$  deduced from the compatibility relation (6). Figure 2 clearly shows the influence of the density ratio  $s$  on the spreading rate, even for these incompressible flows. We can see that the spreading rate is well predicted by the  $k$ - $\varepsilon$  model if the density is uniform ( $s = 1$ ). However, performances deteriorate when  $s \neq 1$ , and the discrepancies on computed  $\delta_0$  can reach 20%. Moreover, the standard  $k$ - $\varepsilon$  model predicts the wrong trend for the variations of  $\delta_0$  vs  $s$ : We remark that the experimental curve indicates an increase of the spatial growth rate with  $s$  whereas the  $k$ - $\varepsilon$  model gives the opposite behavior. Values of  $Kf$  are determined to minimize the differences, and it can be seen in Fig. 2 that negligible discrepancies between computations and experiments are now obtained. As a matter of fact, the new  $\sigma_\varepsilon$  coefficient forces the incompressible mixing layer to reproduce the subsonic effects; consequently, it controls the density

**Table 2** Characteristics of the validation cases for mixing layers

Parameter	$M_c \approx 0$	$M_c = 0.45$	$M_c = 0.65$	$M_c = 1.00$
$s$	$\frac{1}{7}, 1, 7$	$\frac{1}{7}, 7$	$\frac{1}{7}, 7$	$\frac{1}{7}, 7$
$r$	0.65	0.65	0.65	0.65

**Fig. 2** Comparisons of the incompressible spreading rates,  $r = 0.65$ .

influence in a consistent way. In the following, values indicated in Table 1 will be utilized. In Table 2, the value of  $Kf$  for  $s = 1$  has been determined to be  $\sigma_\varepsilon = 1.3$ . The values for  $s = \frac{1}{7}$  and 7 have been obtained numerically to recover the correct variations of the spreadingrate. In such conditions,  $\sigma_\varepsilon$  ranges from 0.53 (for  $s = 7$ ) to 4.33 (for  $s = \frac{1}{7}$ ). As pointed out by Cazalbou et al.,<sup>26</sup> the regularity of the solutions at the outer edges implies some conditions on  $\sigma_k$  and  $\sigma_\varepsilon$ :

$$\sigma_\varepsilon/2 < \sigma_k < 2, \qquad 2\sigma_k - 1 \leq \sigma_\varepsilon$$

The first conditions imply that turbulent quantities vanish at a finite distance, with a balance between transport and diffusion near the edge. The second one is imposed to avoid a velocity profile with an infinite slope at the boundary. For  $s = \frac{1}{7}$ , the first condition is not obeyed, whereas the case  $s = 7$  violates the second one. However, no particular problem appeared with the shape of the velocity profile at the vicinity of the external flows, perhaps because the mesh size is large in this zone. Note that this weakness of the proposed model is not inherent to the formulation of the problem itself (compatibility between coefficients and density gradients in a given flow), but is due to an oversimplification in our application. Relations (5) and (6) are valid only in the vicinity of the maximum of  $k$ , but with the simple assumptions used in the present work, they were applied in the whole section.

C. Compressible Mixing Layer Results

1. Procedure

With the earlier results, we have only considered the influence of density on the spreading rate. Now, it has to be coupled to the Mach number effect. To make comparisons with experimental data, we use the normalized spreading rate defined by  $\delta/\delta_0$  and the Langley consensus curve  $\delta/\delta_0 = \Phi(M_c)$ , which was deduced from a compilation of experimental data.<sup>4</sup> In many numerical works (see Ref. 27, 28, or 29), the incompressible spreading rate  $\delta_0$  is obtained from another computation with the same model at a low convective Mach number and keeping the same values of ratios  $r$  and  $s$ . This value,  $\delta_{0num}$ , was retained for normalization and  $\delta_{num}/\delta_{0num}$  was compared to the Langley curve  $\Phi(M_c)$ . As will be shown in the next section, this procedure is correct provided it has been previously checked that  $\delta_{0num}$  matches correctly the low-speed correlation. Otherwise the coupling with the compressible turbulence model may be misleading and alter the conclusion on the model efficiency.

2. Computations with the Standard  $k-\varepsilon$  Model Associated with the Sarkar–Erlebacher–Hussaini–Kreiss Corrections

The NASA Langley Research Center curve puts in evidence the influence of the convective Mach number on the spreading rate regardless of the other parameters that govern the flow. For a given experiment with spreading rate  $\delta$ , in the NASA Langley Research Center curve representation, the density gradient influence is confined in the normalization by the incompressible spreading rate  $\delta_0$ . Therefore, the sole comparison of numerical results with the NASA Langley Research Center curve is not sufficient to assess the capability of a model to compute accurately a given mixing layer: The discrepancy with the experiment can be totally hidden in the normalization by the incompressible spreading rate  $\delta_0$ .

An example of such a mismatch is given in Figs. 3 and 4. In Fig. 3, the results of computations with the standard  $k-\varepsilon$  model ( $\sigma_\varepsilon = 1.3$ ) associated with  $F(M_t) = 1 + 0.5M_t^2$  are shown. Here  $\delta$  is normalized by the value of the incompressible rate of spread given by the same numerical code. The evolution of  $\delta_{0num}$  is not excellent but gives the expected overall trend. Moreover, the same values are found for  $s = 7$  and  $\frac{1}{7}$ . Figure 4 shows the same results with  $\delta$  normalized by  $\delta_{0exp}$ . It now appears clearly that the results for  $s = \frac{1}{7}$  and 7 do not collapse together: Compressibility is no longer represented by the convective Mach number alone. Moreover, none of the computed trend is satisfactory, and the actual value of the spreading rate can be in error by 30%. In fact, such a model has serious difficulties with accurately computing high-speed mixing layers with density gradient.

Table 3 Comparison of the compressible spreading rates:  $s = \frac{1}{7}$

$M_c$	$\delta_{exp}$ Eq. (7)	$\delta_{num}$ (discrepancy in %)	
		$k-\varepsilon$ ( $\sigma_\varepsilon = 1.3, \alpha = 0.5$ )	rel. compa. ( $\sigma_\varepsilon \neq 1.3, \alpha = 1.66$ )
0.45	0.0291	0.0310 (+6.5)	0.0256 (−12.0)
0.65	0.0224	0.0287 (+28.1)	0.0209 (−6.7)
1.00	0.0146	0.0248 (+69.9)	0.0150 (+2.7)

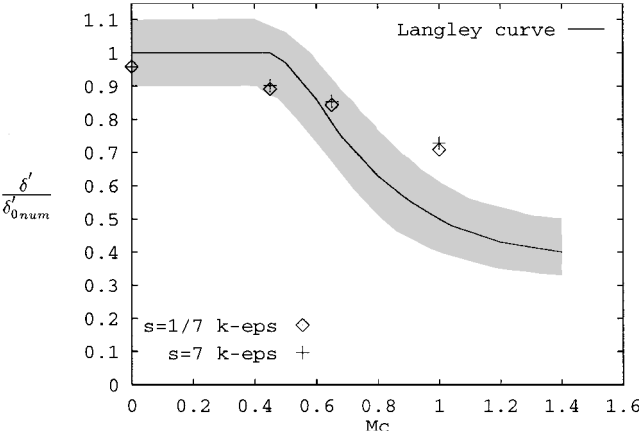


Fig. 3 Compressible spreading rates: results of  $k-\varepsilon$  model with  $\sigma_\varepsilon = 1.3$  and SEHK compressibility model.

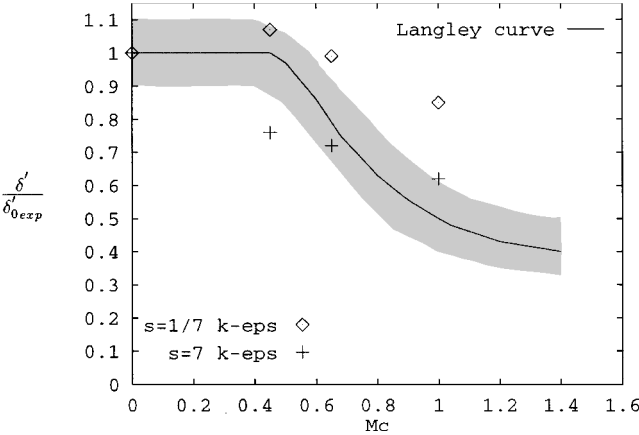


Fig. 4 Compressible spreading rates: results of  $k-\varepsilon$  model.

3. Computations with Variable  $\sigma_\varepsilon$  and Compressibility Corrections

The case with variable  $\sigma_\varepsilon$  and compressible turbulence modeling is now examined. Computations were performed with relation (5) into which the values of  $Kf$  determined by the procedure described in Secs. III.B and V.B have been inserted. These computations were associated with a compressibility model of the Sarkar–Erlebacher–Hussaini–Kreiss type (SEHK):  $F(M_t) = 1 + \alpha M_t^2$ . Because now  $\sigma_\varepsilon$  depends on  $M_c$ ,  $s$ , and  $r$ , it is expected that the value of the constant  $\alpha = 0.5$  has to be revised. It was found that this value predicted the spreading rate for  $s = \frac{1}{7}$  rather poorly, but that the value  $\alpha = 1.66$  gives rather satisfactory results for  $\delta/\delta_0$  (see Table 3 and Fig. 5). In Fig. 5, the spreading rate  $\delta$  is referred to  $\delta_{0exp}$  [relation (7)]; using  $\delta_{0num}$  with this model would have provided practically the same result because  $\delta_{0exp} \sim \delta_{0num}$  (Fig. 2). The determination of  $\alpha$  was made for  $s = \frac{1}{7}$ . However, Fig. 5 shows that computations for  $s = 7$  provide practically the same value for  $\Phi(M_c)$ : The convective Mach number can be used as a compressibility parameter for the spreading rate as found in the experiments. To sum up, note that with the present solution a very sound behavior is found: Discrepancies on spreading rate remain weak for all of the computed Mach numbers, and the

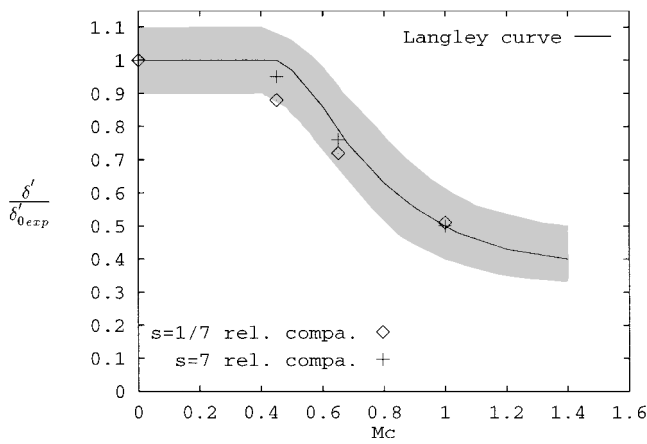


Fig. 5 Compressible spreading rates: results of the modified  $\sigma_\varepsilon$  with  $\alpha = 1.66$ , rel. compa. indicates that the compatibility relation (5) has been used to determine  $\sigma_\varepsilon$ .

results obtained for the two density ratios are very close to each other and remain in agreement with the Langley curve.

## VI. Conclusions

The present work has investigated the performances of the  $k-\varepsilon$  model in compressible mixing layers with high-density variations. It has been shown that the standard  $k-\varepsilon$  model, even if used with a compressibility correction, gives poor predictions of spreading rates; the compressible turbulence modeling can certainly be improved, but the deficiencies increase for compressible mixing layers in the presence of large density ratios. To cure this problem, a relationship redefining the  $\sigma_\varepsilon$  coefficient has been proposed to take into account both the density and the compressibility influence. This relation for  $\sigma_\varepsilon$  ensures compatibility between the density variations and possible compressibility effects. A basic idea of this study was the necessity to have an efficient model for subsonic flows with density variation, to compute supersonic flows properly. We have shown that when this is achieved, convenient results can be obtained for the computation of mixing layers with high-density variation even in supersonic flows. It has also been shown that the values of the constants for the compressible modeling also depend on the density variations and, therefore, cannot be tuned in flows with constant density.

The present attempt can certainly be generalized by allowing a dependence of the  $\sigma_\varepsilon$  coefficient on local variables, instead of defining it in terms of the global characteristics of the mixing layer or by improving the assumptions made to derive the relationship defining  $\sigma_\varepsilon$ . However, this work has shown that compressibility and density gradients are not separable when computed with a  $k-\varepsilon$  model. The prerequisite is to use a model convenient for variable density flows. This is an important requirement because the efficiency of the compressibility correction appears to be very sensitive to this effect. If a correct model for low-speed flows with variable density is used, the proposed conditions between the compressible turbulence model and the constants in the dissipation equation lead to accurate computations of supersonic turbulent shear flows.

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